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Residue Complex for Sklyanin Algebras of Dimension Three

Kaushal Ajitabh*

University of Texas at Austin, Austin, Texas 78712

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We describe the residue complex for three-dimensional Sklyanin algebras, which are the interesting special cases of quantum polynomial rings in three variables. In particular, we show that the multiplicities of the point modules in the residue com-

are *fewer* point modules in the quantum case) by pointing out two quantum anomalies.

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0. INTRODUCTION

The work presented in this paper was motivated by a question of M. Artin on the "residue complex" (minimal injective resolution) for three-dimensional Sklyanin algebras, or more generally for Artin–Schelter regular algebras of dimension 3: a question he sometimes called a "quantum mystery." Regular algebras of dimension 3, with three generators of degree 1, are "quantum" or non-commutative analogues of commutative polynomial rings in three variables [Ar]; and we refer to the case of commutative polynomial rings as classical. Now we explain the question, recalling the classical picture first.

* Current address: Department of Mathematics, Florida International University, University Park, Miami, FL 33199. E-mail: ajitabhk@servms.fiu.edu.

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Let R be a commutative polynomial ring R in three variables (each of degree 1), over an algebraically closed field k. It is well known that a graded minimal injective resolution of R is of the form

$$0 \to R \to I^3 \to I^2 \to I^1 \to I^0 \to 0, \tag{0.1}$$

where using an unusual numbering of the superscripts on I, I^j is a direct sum of indecomposable injective R-modules of Gelfand–Kirilov (gk-) dimension j or, in other words, I^j is a direct sum of injective hulls of gk-j critical R-modules. (A module is gk-j *critical* if it has gk-dimension j and all its proper quotients have lower gk-dimension.) In the language of algebraic geometry, when we think of the sheafified version of (0.1) in terms of a residue complex for the scheme $\mathbf{P}^2_k$, the term I^j corresponds to what are called "dual modules" associated with codimension $3-j$ points of $\mathbf{P}^2_k$ (we denote the dimension of the empty scheme by -1), see, for example, [Ye1]. In a more concrete way, the resolution (0.1) is of the form

$$0 \to R \to Q_R \to \bigoplus_{\phi} E(R/\phi R) \to \bigoplus_{p} E(N_p) \to R'(3) \to 0. \tag{0.2}$$

Here Q_R is the graded field of fractions of R and is an injective hull of the gk-3 critical R-module R. The third term is the direct sum of injective hulls $E(R/\phi R)$ of gk-2 critical R-modules $R/\phi R$, where ϕ runs through the set of irreducible homogeneous elements of R of positive degree, modulo a scalar multiple. The fourth term is the direct sum of injective hulls $E(N_p)$ of gk-1 critical R-modules $N_p = R/(W_p)$ for closed points p of $\mathbf{P}^2_k$, where (W_p) is the ideal of p generated by the two-dimensional space W_p of linear forms in R vanishing at p. We call the modules N_p "point modules." The final term in (0.2) is the shift of R' by 3, where $R' = \mathrm{Hom}_k(R, k)$ is the (Matlis-) dual module of R and is an injective hull of the gk-0 critical R-module k.

Now we consider the quantum case. Regular algebras of dimension 3 [AS, ATV1, ATV2], having three generators of degree 1 and three quadratic relations, are viewed as "twists" or "quantizations" of the commutative polynomial ring $k[x, y, z]$. Artin also calles them *quantum polynomial rings* in three variables [Ar]. They have geometric realizations in terms of twists by certain automorphism σ of a certain scheme E, which is either a cubic divisor in the projective plane or the projective plane itself [ATV1]. In the case when E is a cubic divisor, we call the algebra an elliptic algebra. One of the most interesting cases of an elliptic algebra is the one in which the cubic divisor E is an elliptic curve and the automorphism σ is a translation by a point on E. In this case the algebra is called a (three-dimensional) Sklyanin algebra. (These and higher-dimensional Sklyanin algebras are studied also in [OF, Sk].) In this paper we are interested only in the Sklyanin case of a quantum polynomial ring, and from here on we refer to this as the *quantum case*, in contrast to the classical case of a commutative polynomial ring in three variables. Let A be a Sklyanin algebra and assume that the automorphism σ has finite order. It is known then [ATV1, ATV2] that the critical A-modules of gk-dimension 1, up to shifts, are again the point modules $N_p = A/W_pA$ where now p runs over only the cubic divisor E and not the whole projective plane. So, there are "fewer" point modules in the quantum case. Now let

$$0 \to A \to E^3 \to E^2 \to E^1 \to E^0 \to 0 \tag{0.3}$$

be a graded minimal injective resolution of A as a left or right A-module, which we will simply refer to as a *residue complex for* A. One would expect that as in the classical case, the E^1 term would be a direct sum of injective hulls of point modules. But, as there are much fewer point modules in the quantum case, this brings us to the "quantum mystery." The question is what makes it possible to have a resolution of A with so *fewer* points modules, when it is known that the homological properties of quantum polynomial rings in three variables are very similar to those of the commutative ones [ATV2]. One suspects whether the multiplicities of the point modules in the E^1 term would possibly be greater than one, perhaps even infinite!

We answer this question by showing that the multiplicities of the injective hulls of point modules are all 1, just as in the commutative case. In fact, we describe all the terms in the resolution. Using the notations of [ATV2], let g be a normalizing element of degree 3 in A (unique up to a scalar), B the ring $A/(g)$, and $\Lambda = A[g^{-1}]$ the $\mathbf{Z}$-graded ring obtained by adjoining g^{-1} to A. Then, our main result (Theorem 4.1) is:

THEOREM 1. *A residue complex* (0.3) *for a three-dimensional Sklynanin algebra A is of the form*

$$0 \to A \to Q_A \to \frac{Q_A}{\Lambda} \oplus \frac{Q_A}{A_{(g)}} \to \bigoplus_p E(N_p) \to A'(3) \to 0, \tag{0.4}$$

where Q_A is the graded field of fractions of A, $A_{(g)}$ is the graded localization of A at (g), $E(N_p)$ is an injective hull of the point module N_p, the direct sum in the fourth term is over the points of the elliptic curve associated to A, and $A' = \mathrm{Hom}_k(A, k)$ is the Matlis-dual of A.

The reason why we do have a residue complex for A with fewer point modules and still with the same multiplicity as in the classical case has to do with the fact that the ring $\Lambda = A[g^{-1}]$ has global dimension 1 and not 2, as its commutative counterpart would have (see Section 5). We observe that there are two *quantum anomalies* (by an anomaly we mean a result different from its classical analogue): one is that in the quantum case there are no points outside the cubic divisor, and the other is that the global dimension of "the open complement" Λ differs from its classical analogue by 1. These two anomalies, in a way, "cancel" each other and thus we get a resolution of the quantum polynomial ring A which has same multiplicities of the point modules as that of the commutative polynomial ring R has.

The proof of the main theorem hinges upon two things: first is a rather explicit construction of the resolution up to the E^2 term and the second, and the more crucial, is a result about the purity of resolution:

THEOREM 2. *E^j are pure A-modules of* gk-*dimension j ($j = 0, 1, 2, 3$).*

(A non-zero module M is called pure if every non-zero submodule of M has same gk-dimension as M.)

While our main interest is in studying the residue complex, we are quite naturally led to study the indecomposable injective modules in general. The structure of the paper is as follows. In the first section, we collect the standard results and define the notation of gk-j *equivalence* and *pure algebra*, which are related to the nature of indecomposable injective modules. In the second section, our main objective is in constructing the resolution of A up to the E^2 term. In the third section, we prove the theorem on the purity of resolution (Theorem 3.2) using an analysis of Ext groups and gk-dimensions of various modules and show that A is a pure algebra. In the fourth section, we prove our main result on the residue complex and, in particular, compute the multiplicities of point modules. The fifth section gives a comparison between the quantum and classical cases.

1. PRELIMINARIES

This section is devoted primarily to fixing the notations and collecting the basic results. We keep the discussion more general in the first few paragraphs.

(1.1) Let k be a fixed field. Unless otherwise specified, R will always denote an arbitrary N-graded connected Noetherian k-algebra, finitely generated in degree 1, and satisfying the polynomial growth condition. Here connected means $R_0 = k$, and the polynomial growth condition means that every finitely generated graded left of right R-module M as a Hilbert polynomial: there exists $p_M \in \mathbf{Q}[x]$, say $p(x) = c_0 + c_1 + \cdots + c_d x^d$ ($c_d \neq 0$), such that $\dim_k M_n = p(n)$ for $n >> 0$. Then the Gelfand–Kirillov (gk-) dimension of M, denoted by $\mathrm{gk}(M)$, is $d+1$ and the multiplicity (also called the Bernstein number) of M, denoted by $e(M)$, is $(d!)\,c_d$. The numbers $\mathrm{gk}(M)$ and $e(M)$ are intergers with $\mathrm{gk}(M) \geqslant -1$, $e(M) > 0$ (the zero modue has gk-dimension -1, by definition). For an arbitrary module, not necessarily finitely generated, the gk-dimension is the supremum of the gk-dimensions of its finitely generated submodules. A module of gk-dimension m will simply be called a gk-m *module*. We denote by R° the opposite algebra of R and by R^e the algebra $R \otimes_k R^\circ$, so a left R-module is a right R°-module and an (R, R)-bimodule is a right R^e-module. Let $\mathrm{Gr} - R$ (resp. $\mathrm{gr} - R$) denote the category of all graded right R-modules (resp. the category of finitely generated graded right R-modules) with morphisms being R-homomorphisms of degree 0. There is a diagram of restriction functors

$$\begin{array}{ccc} \mathrm{Gr} - R^e & \xrightarrow{\mathrm{Res}_{R^\circ}} & \mathrm{Gr} - R^\circ \\ {\scriptstyle \mathrm{Res}_R}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_k} \\ \mathrm{Gr} - R & \xrightarrow[\mathrm{Res}_k]{} & \mathrm{Gr} - k \end{array} \tag{1.2}$$

in which all the functors are exact, and the functors Res_R and Res_{R° map injectives to injectives [Ye2, 2.1]. By the term R-module we will mean a right or left graded R-module, and by finite R-module a finitely generated R-module. For an R-module M, $E_R(M)$ (or simply $E(M)$, if there is no confusion) denotes graded injective hull of M. For (R, R)-bimodules M and E, we say that E is an injective hull of M in $\mathrm{Gr} - R$ and in $\mathrm{Gr} - R^\circ$ if $\mathrm{Res}_R(E)$ is an injective hull of $\mathrm{Res}_R(M)$ in $\mathrm{Gr} - R$ and $\mathrm{Res}_{R^\circ}(E)$ is an injective hull of $\mathrm{Res}_{R^\circ}(M)$ in $\mathrm{Gr} - R^\circ$. More often we will write $R - \mathrm{Gr}$ instead of $\mathrm{Gr} - R^\circ$. From here on, the restriction functors will be implicit, for the sake of legibility. The gk-dimension is an exact dimension function on $\mathrm{gr} - R$, i.e., for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathrm{gr} - R$, $\mathrm{gk}(X) = \max\{\mathrm{gk}(X'),\ \mathrm{gk}(X'')\}$. Recall that a non-zero R-module M is called *critical* if every proper quotient of M has lower gk-dimension, it is called *pure* if all its non-zero submodules have the same gk-dimension as M, and it is called *uniform* if it does not contain a direct sum of proper submodules (or, equivalently, every non-zero submodule of M is essential in M). An injective module is uniform if and only if it is non-zero and indecomposable, so the terms "uniform injective" and "non-zero indecomposable injective" are the same [GW, 4.10], and we will generally use the former. Some standard facts, of repeated use to us, about critical, pure, uniform, and indecomposable injective modules are collected in the next proposition with a sketch of some of the proofs.

PROPOSITION 1.3. *Let R be a ring as in* (1.1). *In* (ii)–(viii) *of the following all modules are R-modules.*

(i) *Let $\bar{R}$ be a quotient ring of R (by a two-sided ideal). An $\bar{R}$-module is critical, pure, or uniform as an $\bar{R}$-module if and only if it is so as an R-module.*

(ii) *A non-zero submodule of a critical (resp. pure, uniform) module is critical (resp. pure, uniform). A finite direct sum of* gk-*m pure modules is* gk-*m pure.*

(iii) *A critical module is pure and uniform.*

(iv) *Every non-zero finite module M contains a critical submodule (but, M does not necessarily contain a critical submodule of the same* gk-*dimension as M, see Proposition* 1.6).

(v) *Uniform injective modules are exactly injective hulls of finite critical modules.*

(vi) *A non-zero homomorphism from a critical module M_1 to a pure module M_2 of equal* gk-*dimensions, say m, is necessarily injective. Further, if the cokernel has* gk-*dimension less than m then M_2 is critical.*

(vii) *If R is a domain, then R is a critical R-module. Further, every critical R-module of the same* gk-*dimension as R has an R-submodule isomorphic to $R(-n)$ for some n, and one can take $n \geqslant 0$ if* $\mathrm{gk}(R) > 0$.

(viii) *A* gk-*m essential extension of a* gk-*m pure module is pure.*

Proof. The results (i)–(iv) are standard, and (v) follows from (iii) and (iv): A uniform injective module E is an injective hull of every one of its non-zero submodules and certainly E contains a finite module, hence by (iv) a finite critical module; and conversely, since an injective hull of a uniform module is uniform, so is an injective hull of a critical module by (iii).

We now prove (vi). Let K be the kernel of a given non-zero homomorphism from M_1 to M_2. If K is non-zero then M_1/K has gk-dimension $<m$ because M_1 is critical, but as M_1/K injects into M_2 this gives a contradiction because M_2 is pure of gk-dimension m. This proves the first part. Now identify M_1 with its image in M_2 and assume $\mathrm{gk}(M_2/M_1) < m$. Let X be an arbitrary non-zero submodule of M_2. First note that $M_1 \cap X \neq 0$, else X would inject into M_2/M_1 and we would get a contradiction $m = \mathrm{gk}(X) \leqslant \mathrm{gk}(M_2/M_1) < m$. Now the following exact sequence shows that $\mathrm{gk}(M_2/X) < m$, thus completing the proof that M_2 is critical:

$$0 \to M_1/(M_1 \cap X) \to M_2/X \to M_2/(M_1 + X) \to 0.$$

The first part of (vii) is standard. To see the second part of (vii), let M be a critical R-module of the same gk-dimension as R. In view of (vi), it suffices to note that there is a non-zero R-homomorphism $R(-n) \to M$

given by multiplication by a non-zero homogeneous element x of M of degree n, and we can taken $n \geqslant 0$ if $\mathrm{gk}(M) = \mathrm{gk}(R) > 0$.

To prove (viii), let $M_1 \hookrightarrow M_2$ be an essential extension with $\mathrm{gk}(M_1) = \mathrm{gk}(M_2) = m$, and M_1 pure. Supposing that X were a non-zero submodule of M_2 with $\mathrm{gk}(X) < m$, we would have a non-zero submodule $M_1 \cap X$ of M_1 with $\mathrm{gk}(M_1 \cap X) \leqslant \mathrm{gk}(X) < m$, contradicting the purity of M_1. ∎

(1.4) It is standard [GW, 4ZC] that for R-modules M_1 and M_2, we have $E_R(M_1) \simeq E_R(M_2)$ if and only if there exist essential R-submodules $P_1 \subset M_1$ and $P_2 \subset M_2$ such that $P_1 \simeq P_2$. Now if $E_R(M_1) \simeq E_R(M_2)$ for *critical* R-modules M_1 and M_2 say of gk-dimensions m_1 and m_2, respectively, then P_1 and P_2 (as in the previous statement) would have gk-dimensions m_1 and m_2, respectively, because M_1, M_2 being critical are pure. It follows that $m_1 = m_2$. This shows that we can classify uniform injective R-modules into classes $\mathscr{E}^j$ such that *a uniform injective R-module E is of class $\mathscr{E}^j$* if and only if $E = E_R(M)$ for some finite critical R-module M of gk-dimension j. A uniform injective module of class $\mathscr{E}^j$ may not necessarily have gk-dimension j (see the example following Proposition 1.6).

Definition 1.5. *Let R be an algebra as in* (1.1). *We say that R is a pure algebra if all uniform injective R-modules are pure.*

Note that R being pure as an algebra is different from R being a pure R-module. Now we have the following equivalent formulations of R being a pure algebra.

Proposition 1.6. *Let R be an algebra as in* (1.1). *The following conditions are equivalent*

(i) *R is a pure algebra*;

(ii) *Every finite R-module of gk-dimension j contains a finite critical R-submodule of* gk-*dimension j*;

(iii) *The injective hull of any R-module of* gk-*dimension j has* gk-*dimension j*;

(iv) *A uniform injective R-module of class $\mathscr{E}^j$ has* gk-*dimension j*,

Proof. We show (i) ⇒ (ii). Let M be a finite gk-j module. We can write its injective hull E as $E = \bigoplus_\nu E^\nu$, where E^ν is a direct sum of uniform injective modules pure of gk-dimension ν. We have $E^\nu \cap M \neq 0$ for all ν, because M is essential in E. Now E^ν being gk-ν pure (1.3)(ii), we have $\nu = \mathrm{gk}(E^\nu) = \mathrm{gk}(E^\nu \cap M) \leqslant \mathrm{gk}(M) = j$. So $\nu \leqslant j$ for all ν and, as $\mathrm{gk}(E) \geqslant j$, there must be a ν equal to j. Thus there is a finite critical gk-j submodule X of E and then $X \cap M$ is a non-zero finite critical gk-j submodule of M. To show

(ii) $\Rightarrow$ (iii), let M again be a finite gk-j module and E its injective hull. We need to show that for any non-zero finite submodule X of E, $\mathrm{gk}(X) \leqslant j$. Let X be a critical submodule of X of the same gk-dimension as X. Then $Y \cap M$ being a non-zero submodule of Y has the same gk-dimension as Y, thus $\mathrm{gk}(X) = \mathrm{gk}(Y \cap M) \leqslant \mathrm{gk}(M) = j$. As (iii) $\Rightarrow$ (iv) is trivial, we only need to show (iv) $\Rightarrow$ (i), which is immediate in view of (1.3)(viii). ▮

Remark. It follows from (1.3)(i) and (1.6) that the quotient of a pure algebra (by a two-sided ideal) is a pure algebra. In particular, every commutative algebra R (as in (1.1)) is pure because it is a quotient of a commutative polynomial algebra over k, and the latter indeed is known to be pure. We will see in Section 4 that a three-dimensional Sklyanin algebra is pure and how this relates to its residue complex. The following example (due to S. P. Smith) shows that there are non-commutative algebras which are not pure.

EXAMPLE. Let $R = H(\mathrm{sl}(2))$ be the homogenization of the enveloping algebra $\mathscr{U}(\mathrm{sl}(2))$. Thus R is generated over k by homogeneous elements x_0, x_1, x_2, x_3 (each of degree 1), with x_0 central and $R/(x_0 - 1) \simeq \mathscr{U}(\mathrm{sl}(2))$. We show that R is not a pure algebra by showing (1.6)(iii) that there exists a gk-1 module N which has a gk-2 essential extension M.

Identify $\mathscr{U}(\mathrm{sl}(2)) \simeq k[\partial, x\partial, x^2\partial]$, and let $k[x]$ be the left $\mathscr{U}(\mathrm{sl}(2))$-module with the usual action of ∂, $x\partial$, and $x^2\partial$. Now define a graded left R-module $X = k[x] \otimes_k k[x_0]$ with $\deg(x) = 0$, $\deg(x_0) = 1$, and the action $a.(f \otimes x_0^i) = (\bar{a}.f) \otimes x_0^{i+1}$ for $a \in R_1$. Here $\bar{a}$ is the image of a under $R \to k[\partial, x\partial, x^2\partial]$ which maps $x_0 \mapsto 1$, $x_1 \mapsto \partial$, $x_2 \mapsto x\partial$, $x_3 \mapsto x^2\partial$. Let M be the R-submodule of X defined as $M = \sum_{j=0}^{\infty} \sum_{i \leqslant j} kx^i \otimes x_0^j$. The Hilbert function of M is $h_M(n) = n+1$ $(n \geqslant 0)$ and so $\mathrm{gk}(M) = 2$. Now let N be the cyclic R-submodule of M generated by $(1 \otimes 1)$, so $N = R(1 \otimes 1) = 1 \otimes k[x_0]$. The Hilbert function of N is $h_N(n) = 1$ $(n \geqslant 0)$, and so $\mathrm{gk}(N) = 1$. Further, $N \hookrightarrow M$ is essential because for any $f \otimes x_0^i$ $(f \in k[x], \deg f > 0)$, we have $R(f \otimes x_0^i) \cap N \neq 0$.

(1.7) Let R have gk-dimension d. The gk-dimension gives a filtration on $\mathrm{gr} - R$,

$$\mathbf{0} \subset (\mathrm{gr} - R_0) \subset (\mathrm{gr} - R)_1 \cdots \subset (\mathrm{gr} - R)_d = \mathrm{gr} - R, \tag{1.8}$$

where $(\mathrm{gr} - R)_j$ is the full subcategory of $\mathrm{gr} - R$ consisting of R-modules of gk-dimension at most j $(0 \leqslant j \leqslant d)$, and $\mathbf{0}$ is the subcategory consisting only of the zero object. It follows from the gk-dimension being an exact dimension function on $\mathrm{gr} - R$ that for all $j = 0, ..., d$, $(\mathrm{gr} - R)_j$ is *dense* (or thick) in $\mathrm{gr} - R$, and hence dense in $(\mathrm{gr} - R)_i$ for $i \geqslant j$. (A full subcategory $\mathscr{A}$ of an abeian category $\mathscr{C}$ is said to be dense in $\mathscr{C}$ if for any exact sequence

$0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, X is an object of $\mathscr{A}$ if and ony if both X' and X'' are objects of $\mathscr{A}$.) Thus the quotient category $\mathrm{gr}-R/(\mathrm{gr}-R)_j$ is defined for all j [Po, Theorem 3.3].

DEFINITION 1.9. *For the objects M_1 and M_2 of* $\mathrm{gr}-R$, *we say that M_1 is* gk-j *equivalent to M_2 if the canonical projection of M_1 in the quotient category* $\mathrm{gr}-R/(\mathrm{gr}-R)_j$ *is isomorphic to that of M_2.*

This obviously defines an equivalence relation on the objects of $\mathrm{gr}-R$.

Note 1.10. (i) Let Σ_j be the class of all morphisms s in $\mathrm{gr}-R$ such that $\ker(s)$ and $\mathrm{coker}(s)$ are in $(\mathrm{gr}-R)_j$. For R-modules M_1 and M_2 to be gk-j equivalent it is necessary and sufficient that there be an R-module M and morphisms $s_1\colon M_1 \to M$ and $s_2\colon M_2 \to M$ in Σ_j [Po, Sect. 4.3]. In particular, in view of (1.3)(vi), if there is a non-zero morphism between two gk-j critical R-modules then they are gk-$(j-1)$ equivalent. (ii) The category $\mathrm{gr}-R/(\mathrm{gr}-R)_0$ is the same as $\mathrm{proj}-R$ of Artin and Zhang [AZ].

PROPOSITION 1.11. *There is a one-to-one correspondence between*

(i) *isomorphism classes of uniform injective R-modules of class $\mathscr{E}^j$, and*

(ii) gk-$(j-1)$ *equivalence classes of finite critical R-modules of* gk-*dimension j.*

Proof. In view of (1.3)(v), we only need to show that for gk-j finite critical R-modules M_1 and M_2, we have $E(M_1) \simeq E(M_2)$ if and only if M_1 and M_2 are gk-$(j-1)$ equivalent. Now if $E(M_1) \simeq E(M_2)$ then there exist essential submodules $P_\nu \subset M_\nu$ $(\nu = 1, 2,)$ such that $P_1 \simeq P_2$. Now $\mathrm{gk}(M_\nu/P_\nu) < j$, so P_ν is gk-$(j-1)$ equivalent to M_ν which combined with $P_1 \simeq P_2$ implies that M_1 and M_2 are gk-$(j-1)$ equivalent too. Conversely, if M_1 and M_2 are gk-$(j-1)$ equivalent then there exist a gk-j R-module M and morphisms $s_\nu\colon M_\nu \to M$ in $\Sigma_{(j-1)}$ (Note 1.10(i)). Now M_ν being critical and $\mathrm{gk}(\ker s_\nu) < j$ we see that the s_ν are injective. Then the intersection P of the images of M_1 and M_2 in M is non-zero and taking the inverse images of P in M_ν provides essential submodules $P_\nu \subset M_\nu$ with $P_1 \simeq P_2$. Thus $E(M_1) \simeq E(M_2)$ by (1.4). ∎

COROLLARY 1.12. *Suppose R is a domain with $gk(R) = d > 0$ and let Q_R be the graded field of fractions of R. Then, up to an isomorphism, the R-module Q_R is the only uniform injective R-module of class $\mathscr{E}^d$.*

Proof. By (1.3)(vii) and (1.10)(i), every finite critical R-module of gk-dimension d is gk-$(d-1)$ equivalent to $R(-n)$ for some n. But $R(-n)$

being clearly gk-$(d-1)$ equivalent to R, we see that up to gk-$(d-1)$ equivalence R is the only gk-d finite critical R-module. Thus the corollary follows from the proposition, as Q_R is an injective hull of R, in $\mathrm{Gr}-R$ and $R-\mathrm{Gr}$. ▮

We now specialize to three-dimensional Sklynanin algebras which are interesting special cases of regular algebras of dimension 3. Recall [ATV1] that to a regular algebra of dimension 3, with three generators of degree 1, there corresponds a triple $(E, \sigma, \mathscr{L})$ where E is a subscheme of the projective plane $\mathbf{P}_k^2$, σ is an automorphism of E, and $\mathscr{L}$ is an invertible sheaf on E. The case when E is a cubic divisor is called the elliptic case.

DEFINITION 1.13. *By a (three-dimensional) Sklyanin algebra we will always mean the elliptic case of a regular algebra of dimensions* 3 *with three generators of degree* 1 *which satifies the following two hypotheses*: (1) *The associated cubic divisor E is an elliptic curve*, (2) *No positive integral power of σ fixes the class $[\mathscr{L}]$ in* $\mathrm{Pic}(E)$; *i.e., the index s_0 of the algebra A* [ATV2, *Sect.* 7] *is infinite.*

The symbol A will always denote a Sklyanin algebra. We always assume that the base field k is algebraically closed. There is a normalizing element g of degree 3 in A which is unique up to a scalar [ATV1]. Let B be the $\mathbf{N}$-graded ring $A/(g)$, $\Lambda = A[g^{-1}]$ the $\mathbf{Z}$-graded ring obtained by adjoining the inverse of g, and Λ_0 the subring of Λ consisting of degree zero elements. (As pointed out in [ATV2], one can view the normalizing element g as the analogue of the cubic equation defining E, the ring B as the analogue of the homogeneous coordinate ring of E, and Λ_0 as the analogue of the ring describing the "open complement" of E in the plane.) The first hypothesis in (1.13) has the consequence that B is a domain. The second hypothesis has the consequence that every A-module of gk-dimension at most 1 is annihilated by some power of g, and that an A-module of gk-dimension 1 is critical if and only if it is a shifted point module [ATV2, Sect. 7]. The relation between g-torsion and gk-1 modules can be summarized in categorial terms as follows.

(1.14) Let $\mathrm{tor}-g$ be the full subcategory of $\mathrm{gr}-A$ consisting of the g-torsion modules. (An A-module M is said to be g-torsion if every element of M is annihilated by some power of g.) Now $\mathrm{tor}-g$ is also a dense abelian subcategory of $\mathrm{gr}-A$, so the quotient category $\mathrm{gr}-A/\mathrm{tor}-g$ is defined. Since every A-module of gk-dimension at most 1 is g-torsion, the canonical projection functor $P\colon \mathrm{gr}-A \to \mathrm{gr}-A/\mathrm{tor}-g$ factors through $\mathrm{gr}-A/(\mathrm{gr}-A)_1$. Further, let $\mathrm{gr}-\Lambda$ be the category of graded finite Λ-modules, and $T\colon \mathrm{gr}-A \to \mathrm{gr}-\Lambda$ the canonical localization functor $M \rightsquigarrow M \otimes_A \Lambda$. Then $\mathrm{tor}-g$ is the kernel of the functor T and the category

$\text{gr}-\Lambda$ is equivalent to $\text{gr}-A/\text{tor}-g$ [Po, Theorem 49]. Finally, as Λ is strongly graded [ATV2, 7.4], the category $\text{gr}-\Lambda$ is equivalent to the category $\text{mod}-\Lambda_0$ of finite Λ_0-modules [NV, Chap. A, I.3.4]. Thus we have the following commutative diagram:

$$\begin{tikzcd} & \text{gr}-A \arrow[r] \arrow[d, "(\cdot)\otimes_A \Lambda"'] \arrow[dr, "P"] & \text{gr}-A/(\text{gr}-A)_1 \arrow[d, dotted] \\ \text{mod}-\Lambda_0 & \text{gr}-\Lambda \arrow[l, "\sim"'] \arrow[r, "\sim"] & \text{gr}-A/\text{tor}-g \end{tikzcd} \tag{1.15}$$

(1.16) For a three-dimensional Sklyanin algebra A, the uniform injective A-modules will be of the classes $\mathscr{E}^j$ with $j=0, 1, 2, 3$.

(i) The only critical A-modules of gk-dimension 0 are the shifts of the simple module k. It is standard that the (A, A)-bimodule $A' = \text{Hom}_k(A, k)$ is an injective hull of k both in $\text{Gr}-A$ and $A-\text{Gr}$. Thus the only uniform injective A-modules of class $\mathscr{E}^0$ are the shifts of A'. It is clear that the gk-dimension of A' is 0.

(ii) The critical A-modules of gk-dimension 1 are precisely the shifts $N_p(n)$ of the point modules N_p [ATV2]. It is known [ATV2, Sect. 6; Aj1, Sect. 2] that $N_p(n)$ is gk-0 equivalent to $N_{\sigma^n p}$ for all $n \in \mathbf{Z}$, and that N_p is *not* gk-0 equivalent to N_q for $p \neq q$. Thus, it follows from Proposition 1.11 that, up to an isomorphism, the uniform injective A-modules of class $\mathscr{E}^1$ are exactly the injective hulls $E_A(N_p)$ of normalized (unshifted) point modules.

(iii) A being a domain, the description of uniform injective A-modules of class $\mathscr{E}^3$ is trivial (Corollary 1.12): the $\mathbf{Z}$-graded field of fractions Q_A of A is an injective hull of A in $\text{Gr}-A$ and in $A-\text{Gr}$ and, up to an isomorphism, is only uniform injective A-module of class $\mathscr{E}^3$.

We now study uniform injective A-modules of class $\mathscr{E}^2$ in the next section.

2. INJECTIVE MODULES OF gk-DIMENSION 2

In this section, we construct a residue complex (0.3) for a three-dimensional Skylanin algebra up to the E^2 term. We realize the E^2 term as a direct sum of two injective factors, one is g-torsion and the other g-torsion-free. The first few results (some of which are standard) hold under more general conditions.

(2.1) Unless otherwise specified, the symbol R in this section will always denote an algebra as in (1.1) with these additional hypotheses: R is a domain with $\mathrm{gk}(R) > 0$, and it has a homogeneous normalizing element g of positive degree such that the two-sided ideal (g) is completely prime, i.e., $R/(g)$ is also a domain. We denote $R/(g)$ by $\bar{R}$. An element of an R-module M is said to be g-torsion if it is annihilated by some power of g. The g-torsion elements of M form a submodule which we denote by $t_g(M)$. We say M is g-torsion or g-torsion-free accordingly as $t_g(M) = M$ or $t_g(M) = 0$. Let $\Lambda_R = R[g^{-1}]$ be the $\mathbf{Z}$-graded ring obtained by adjoining the inverse of g to R. The ring Λ_R is the same as the left and right ring of quotients with respect to the left and right Ore subset $\{g^n \mid n \geqslant 0\}$. We denote by Q_R the $\mathbf{Z}$-graded field of fractions of R, which is also the graded field of fractions of Λ_R.

(2.2) Recall the following standard facts (see, for example, [MR, Sect. 4.2]). A two-sided ideal J of a right Noetherian ring R is said to have the right *Artin–Rees property* if it satifies the following equivalent conditions:

(i) If I is a right ideal of R then $I \cap J^n \subset IJ$ for some n,

(ii) If M is a finitely generated right R-module and N is an essential submodule of M with $NJ = 0$ then $MJ^n = 0$ for some n,

PROPOSITION 2.3 [MR, 4.2.6]. *Let R be a right Noetherian ring and J an ideal generated by normalizing elements. Then J has the right Artin–Rees property.*

It is straightforward to see that the graded (left and right) versions of (2.2) and (2.3) are true. In particular, applying this to the algebra R of (2.1), we see that the ideal (g) has the left and right Artin–Rees property. This has the following consequence for uniform injective modules.

PROPOSITION 2.4. *A uniform injective R-module is either g-torsion-fee or g-torsion (equivalently, the g-torsion submodule of an injective R-module is injective).*

Proof. Let E be a uniform injective right R-module. If E has a g-torsion element then $E_1 = \{x \in E \mid xg = 0\}$ is a non-zero submodule. Now letting x be an arbitrary element of E we show that $xg^n = 0$ for some n. Put $M = xR$ and $N = M \cap E_1$. As E is uniform, N is non-zero and essential in M. Since $E_1 g = 0$, we have $Ng = 0$. Applying the condition (ii) of (2.2) we see that $Mg^n = 0$, hence $xg^n = 0$, for some $n \geqslant 1$. ∎

In view of (1.3)(v), we get the following corollary.

COROLLARY 2.5. *A finite critical R-module is either g-torsion or g-torsion-free.*

Now observe that the g-torsion-free (resp. g-torsion) uniform injective R-modules are "parametrized" by the ring Λ_R (resp. $\bar{R}$), in the following sense:

PROPOSITION 2.6. (i) *There is a bijective correspondence between the isomorphism classes of g-torsion-free uniform injective R-modules and the isomorphism classes of uniform injective Λ_R-modules. In fact, a g-torsion-free uniform injective R-module is naturally a uniform injective Λ_R-module, and conversely.*

(ii) *There is a bijective correspondence between the isomorphism classes of g-torsion uniform injective R-modules and the isomorphism classes of uniform injective $\bar{R}$-modules. In fact, the g-torsion uniform injective R-modules are exactly the injective hulls over R of finite critical $\bar{R}$-modules.*

Proof. The ring Λ_R being the ring of quotients with respect to the Ore denominator subset $\{g^n \mid n \geqslant 0\}$, the statement (i) is a special case of a general fact [GW, 9.16]: If S is an Ore denominator subset in a ring R then an S-torsion-free injective (resp. uniform injective) R-module is an injective (resp. uniform injective) RS^{-1}-module, and conversely.

To see (ii), let E be a g-torsion uniform injective R-module and let $E_1 = \{x \in E \mid xg = 0\}$. Then E_1 is injective as an $\bar{R}$-module. Further, E_1 being a submodule of uniform R-module E is uniform as an R-module, and hence uniform as an $\bar{R}$-module (1.3)(i). This gives the correspondence in one direction. In the other direction, for a uniform injective $\bar{R}$-module (1.3)(i), $E_R(I)$ is a uniform R-module. Also $E_R(I)$ is g-torsion because it contains a g-torsion submodule I (2.4). It is easy to see that these two correspondences define a bijection on the level of isomorphism classes. ∎

COROLLARY 2.7. *Let* $\mathrm{gk}(R) = d$. *Then $\bar{R}$ is a critical R-module of* gk*-dimension $d-1$ and, up to an isomorphism, $E_R(\bar{R})$ is the only g-torsion uniform injective R-module of class $\mathscr{E}^{d-1}$. In other words, if M is a finite critical R-module of* gk*-dimension $d-1$ then the following conditions are equivalent*: (i) *M is g-torsion*, (ii) *M is* gk-$(d-2)$ *equivalent to $\bar{R}$ as R-modules*, (iii) $E_R(M) \simeq E_R(\bar{R})$, (iv) *M contains an R-submodule isomorphic to $\bar{R}(-n)$ for $n \geqslant 0$.*

Proof. First it is clear from a consideration of the Hilbert polynomials that $\mathrm{gk}(\bar{R}) = d-1$. Now as $\bar{R}$ is a domain the rest of the statement follows from a combination of (1.3)(i), (1.3)(vii), (1.12), and (2.6). ∎

We now give a construction of $E_R(\bar{R})$ using localization of R at the ideal (g). Define

$$S = \{s, \text{ homogeneous element of } R \mid s \notin (g)\}. \tag{2.8}$$

Since $\bar{R} = R/(g)$ is a domain, we see that S is a multiplicative subset.

LEMMA 2.9. $\bigcap_{n=1}^{\infty}(g)^n=0$. *Consequently, a homogeneous element* $a\in R$ *can be written as* $a=g^n s$ *with a unique* $n\geqslant 0$, *and a unique* $s\in S$.

Proof. Let r be the degree of g. Then first assertion follows simply by noting that $(g)^n\subset R_{\geqslant rn}$ and $\bigcap_n R_{rn}=0$. To see the second assertion, let a be a given homogeneous element of R. If $a\in S$ then take $n=0$ and $s=a$, else choose the greatest integer n such that $a\in(g)^n$ and so $a=g^n s$ for some $s\in S$. Uniqueness of this representation is evident, as R is a domain. ∎

PROPOSITION 2.10. *S is a left and right Ore subset in R.*

Proof. We check the right Ore condition. Let $0\neq a\in R$ and $s\in S$ be homogeneous elements. Since R is a domain, we have $aR\cap sR\neq 0$. Thus there exist non-zero homogeneous elements b, $t\in R$ such that $at=sb$. Let $t=t'g^m$, $b=b'g^m$ where $t', b'\in S$ and $m, n\geqslant 0$ (Lemma 2.9). As $s\in S$, we have $n\geqslant m$. So cancelling a common factor of g^m from the previous equality we can write $at'=sb'g^{n-m}$, which verifies the right Ore condition. The left Ore condition is similarly verified. ∎

(2.11) The S-torsion in R, i.e., $\{a\in R \mid as=0 \text{ for some } s\in S\}$ is zero, as R is a domain. Thus, S is a denominator subset. We denote by $R_{(g)}$ the $\mathbf{Z}$-graded localization of R with respect to S. The canonical map $R\to R_{(g)}$ is injective and $R_{(g)}$ is a domain with the same field of fractions Q_R as that of R. Further, $(g)\,A_{(g)}$ is a two-sided principal ideal which we will denote simply by (g), if there is no possibility of confusion. A non-zero (resp. non-zero homogeneous) element of Q_R can be written as $g^n uv^{-1}$ for a unique $n\in\mathbf{Z}$ such that $u\in R-(g)$, $v\in S$ (resp. $u, v\in S$). This defines a discrete valuation on Q_R, which we call the *g-valuation*. The ring $R_{(g)}$ is the valuation ring of Q_R for this valuation, and the principal ideal $(g)\,R_{(g)}$ is the unique maximal ideal of $R_{(g)}$. The simple $R_{(g)}$-module $M=R_{(g)}/(g)\,R_{(g)}$ has a projective resolution $0\to(g)\,R_{(g)}\to R_{(g)}\to M\to 0$. Thus we see that the global dimension of $R_{(g)}$ is 1.

PROPOSITION 2.12. *The* (R, R)-*bimodule* $Q_R/R_{(g)}$ *is an injective hull of* $\bar{R}$ *in* Gr-R *and in* R-Gr.

Proof. As the global dimension of $R_{(g)}$ is 1, the exact sequence

$$0\to R_{(g)}\to Q_R\to Q_R/R_{(g)}\to 0 \tag{2.13}$$

shows that $Q_R/R_{(g)}$ is an injective module in Gr-$R_{(g)}$ and in $R_{(g)}$-Gr, and therefore an injective module in $\mathrm{Gr}-R$ and in $R-\mathrm{Gr}$. Now fix a homogeneous element $t\in S$ of degree equal to the degree of g. Then the right (resp. left) R-linear map $R \to Q_R$ given by left (resp. right) multiplication by $g^{-1}t$ defines an essential monomorphism $\bar{R} = R/(g) \hookrightarrow Q_R/R_{(g)}$ in $\mathrm{Gr} - R$ (resp. $R - \mathrm{Gr}$). ▮

Note 2.14. (i) $R_{(g)} \cap \Lambda_R = R$ (inside Q_R), and (ii) $R_{(g)} \otimes_R \Lambda_R = Q_R$ as (R, R)-bimodules. The first is clear and the second follows from tensoring the exact sequence (2.13) with Λ_R.

PROPOSITION 2.15. *There is an exact sequence of* (R, R)-*bimodules*

$$0 \to R \to Q_R \to \frac{Q_R}{R_{(g)}} \oplus \frac{Q_R}{\Lambda_R}, \tag{2.16}$$

where the first map is the canonical injection and the second is the sum of the canonical projections. Further, the monomorphism $Q_R/R \hookrightarrow Q_R/R_{(g)} \oplus Q_R/\Lambda_R$ *is essential, in* Gr-R *and* R-Gr.

Proof. The exactness of the sequence follows from the fact that $R_{(g)} \cap \Lambda_R = R$, inside Q_R. To prove the second statement, we let X denote the third term in (2.16), and show that for any homogeneous element $x \neq 0$ in X, there exists a homogeneous element $a \in R$ such that $0 \neq xa$ is in the image of Q_R/R in X. (To fix notations, we work in Gr-R; of course, a similar proof works in R-Gr.) Let $x = (\pi_1(q_1), \pi_2(q_2)) \neq (0, 0)$, where q_i are homogeneous elements of Q_R and π_1 and π_2 denote the projections of Q_R on $Q_R/R_{(g)}$ and Q_R/Λ_R, respectively. We need to show that for some homogeneous element $a \in R$, we have $(0, 0) \neq xa = (\pi_1(q), \pi_2(q))$, for a homogeneous $q \in Q_R$. We can write $q_i = g^{-n_i} f_i u_i^{-1}$ $(i = 1, 2)$ where $f_i, u_i \in S$ and $n_i \in \mathbf{Z}$. Suppose first that $\pi_1(q_1) \neq 0$. Using the Ore condition of S we can write $u_1^{-1} u_2 = v_2 v_1^{-1}$ for $v_i \in S$. Then taking $a = u_2 v_1$, we get as required

$$xa = (\pi_1(q_1 u_2 v_1), \pi_2(q_2 u_2 v_1)) = (\pi_1(q), \pi_2(q))$$

with $q = q_1 u_2 v_1$ and $\pi_1(q) \neq 0$, $\pi_2(q) = 0$. Indeed, $\pi_1(q) = \pi_1(q_1 u_2 v_1) \neq 0$ because $\pi_1(q_1) \neq 0$ and $u_2 v_1 \in S$, while $\pi_2(q_2 u_2 v_1) = 0 = \pi_2(q)$ because both $q_2 u_2 v_1 = g^{-n_2} f_2 v_1$ and $q = q_1 u_2 v_1 = g^{-n_1} f_1 v_2$ are in Λ_R.

Next suppose that $\pi_1(q_1) = 0$, so $\pi_2(q_2) \neq 0$. Let $n = \max(n_2, 0)$. Then taking $a = g^n$, we again get as required

$$xa = (0, \pi_2(q_2 g^n)) = (\pi_1(q), \pi_2(q))$$

with $q = q_2 g^n$ and $\pi_1(q) = 0$, $\pi_2(q) \neq 0$. Indeed, $\pi_2(q) = \pi_2(q_2 g^n) \neq 0$ because $\pi_2(q_2) \neq 0$ and Q_R/Λ_R is g-torsion-free, while $\pi_1(q) = 0$ because the g-valuation of $q = q_2 g^n$ is $n - n_2 \geqslant 0$. ▮

We now specialize to a three-dimensional Sklyanin algebra A (Definition 1.13). Since A has a normalizing element g of degree 3 (unique up to a scalar) such that $B = A/(g)$ is a domain, it satisfies the hypotheses of (2.1). Let $A_{(g)}$ be the localization of A at (g), $B = A/(g)$, and $\Lambda_A = A[g^{-1}]$. The next proposition summarizes the information we get about uniform injective A-modules of class $\mathscr{E}^2$, by applying the above general discussion to the case of A. First recall that, in view of Proposition 1.11, the uniform injective A-modules of class $\mathscr{E}^2$ can be parameterized, or more precisely indexed, by the gk-1 equivalence classes α of gk-2 finite critical A-modules, so we can label them as $E_\alpha = E_A(M_\alpha)$ where M_α is a gk-2 finite critical A-module representing the gk-1 equivalence class α.

Proposition 2.17. *A uniform injective A-module E of class $\mathscr{E}^2$ is either g-torsion or g-torsion-free. It is g-torsion (resp. g-torsion-free) if and only if $E = E_\beta$ (resp. $E = E_\alpha$ for $\alpha \neq \beta$) where β denotes the* gk-1 *equivalence class of the A-module B. In fact, the (A, A)-bimodule $Q/A_{(g)}$ is isomorphic to E_β both in* $\mathrm{Gr} - A$ *and* $A - \mathrm{Gr}$. *On the other hand, E_α $(\alpha \neq \beta)$ are, naturally, uniform injective Λ-modules.*

We now use a result in the Sklyanin case (which may not be true in the general setting of R considered above), namely that Λ_A has global dimension 1.

Proposition 2.18. *The ring Λ_A has global dimension* 1. *The (A, A)-bimodule Q_A/Λ_A is a g-torsion-free injective module in* $\mathrm{Gr} - A$ *and* $A - \mathrm{Gr}$. *There is an exact sequence of (A, A)-bimodules*

$$0 \to A \to Q_A \to \frac{Q_A}{A_{(g)}} \oplus \frac{Q_A}{\Lambda_A} \tag{2.19}$$

which is a minimal injective resolution of A, in $\mathrm{Gr} - A$ *and* $A - \mathrm{Gr}$, *up to the E^2 term* (*see* (0.3)).

Proof. The fact that $\Lambda = \Lambda_A$ has global dimension 1 is probably known, but for a lack of reference we sketch the proof. For an A-module M, let M_Λ denote the Λ-module $M \otimes_A \Lambda$. It suffices to show that the projective dimension (pd) of M_Λ is at most one for every finite A-module M. As $M_\Lambda = 0$ when $\mathrm{gk}(M) < 2$, we only need to consider the cases $\mathrm{gk}(M) = 2, 3$. Letting $M^\vee = \mathrm{Ext}_A^{3-\mathrm{gk}(M)}(M, A)$ be the dual of M, there is a canonical map $\mu_M\colon M \to M^{\vee\vee}$ where $\mathrm{gk}(\ker \mu_M) < \mathrm{gk}(M)$ and $\mathrm{gk}(\mathrm{coker}\, \mu_M) \leqslant \mathrm{gk}(M) - 2$ [ATV2, Sect. 4]. Thus if $\mathrm{gk}(M) = 2$ then $\ker \mu_M$ and $\mathrm{coker}\, \mu_M$ are annihilated by Λ and $(M^{\vee\vee})_\Lambda \simeq M_\Lambda$, so $pd(M_\Lambda) = pd((M^{\vee\vee})_\Lambda) \leqslant pd(M^{\vee\vee}$ as $A-$module$) = 1$. Finally, suppose M is finite of gk-dimension 3, and we may assume M critical. Then by (1.3)(vii) there is an exact sequence of A-modules $0 \to A(-n) \to M \to P \to 0$ where $\mathrm{gk}(P) \leqslant 2$. Thus we get an exact sequence of Λ-modules $0 \to \Lambda(-n) \to M_\Lambda \to P_\Lambda \to 0$, implying that $pd(M_\Lambda) \leqslant 1$ because we know $pd(P_\Lambda) \leqslant 1$.

Now the global dimension of Λ being 1, the exact sequence

$$0 \to \Lambda_\Lambda \to Q_\Lambda \to Q_\Lambda/\Lambda_\Lambda \to 0 \tag{2.20}$$

shows that $Q_\Lambda/\Lambda_\Lambda$ is an injective module in $\mathrm{Gr} - \Lambda_\Lambda$ and in $\Lambda_\Lambda - \mathrm{Gr}$, and therefore an injective module in $\mathrm{Gr} - A$ and in $A - \mathrm{Gr}$. Obviously it is g-torsion-free. The last statement now follows from Proposition 2.15. ∎

3. PURITY OF RESIDUE COMPLEX

Let A be a three-dimensional Sklyanin algebra (Definition 1.13) and let

$$0 \to A \to E^3 \to E^2 \to E^1 \to E^0 \to 0 \tag{3.1}$$

be a residue complex, i.e., a minimal injective resolution, of A as a right or left A-module. We prove that the residue complex is pure in the sense that:

THEOREM 3.2. *The A-modules E^j are pure of* gk-*dimension* j $(j = 0, 1, 2, 3)$.

At the end of the section we note that a three-dimensional Sklyanin algebra is a pure algebra (Definition 1.5). In the next section we use Theorem 3.2 to describe E^0 and E^1 in the resolution. We prove Theorem 3.2 in a sequence of lemmas. Recall this simple observation (1.3)(viii), which will be used repeatedly:

LEMMA 3.3. *An essential* gk-*m extension of a* gk-*m pure module is pure.*

We will use the following criterion for checking the gk-dimension.

LEMMA 3.4. *Let M be an A-module. Then*

(i) $\mathrm{gk}(M) < 3$ *if and only if* $M \otimes_A Q = 0$.

(ii) $\mathrm{gk}(M) < 2$ *if and only if* $M \otimes_A A_{(g)} = 0$ *and* $M \otimes_A \Lambda = 0$.

Proof. Part (i) is standard [ATV2, 2.30v]. As both Λ and $A_{(g)}$ annihilate finite critical (and therefore all) A-modules of gk-dimension less than 2, the implication in one direction in (ii) is obvious. Now we prove that $M \otimes_A A_{(g)} = 0$ and $M \otimes_A \Lambda = 0$ force $\mathrm{gk}(M) < 2$. Again we may assume M finite and critical. As $M \otimes_A (A_{(g)} \otimes_A \Lambda) = M \otimes_A Q = 0$ (Note 2.14(ii)), we have $\mathrm{gk}(M) \leqslant 2$ by part (i). Now if $\mathrm{gk}(M) = 2$ then being a

g-torsion finite critical A-module M would contain $B(-n)$ for some n (Corollary 2.7). But then we would have $B(-n) \otimes_A A_{(g)} = 0$, a contradiction. Thus $\mathrm{gk}(M) < 2$. ∎

Also, we will identify $E^3 \simeq Q$ and $E^2 = Q/A_{(g)} \oplus Q/\Lambda$ (Proposition 2.18), where we are dropping the subscript A from Q and Λ.

We start with the exact sequence

$$0 \to A \to E^3 \to E^3/A \to 0. \tag{3.5}$$

Lemma 3.3 applied to $A \hookrightarrow E^3$ implies that E^3 is pure gk-3.

Lemma 3.6. *E^3/A is pure of* gk-*dimension* 2.

Proof. The gk-dimension of E^3/A is at most 2, for $E^3/A \otimes_A Q = 0$ (3.4)(i). Since we clearly have $\mathrm{gk}(E^3/A) \geqslant 2$, it suffices to show that $\mathrm{Hom}_A(X, E^3/A) = 0$ for all finite A-modules X with $\mathrm{gk}(X) \leqslant 1$. Applying $\mathrm{Hom}_A(X,\ )$ to the exact sequence (3.5), and using the facts that $\mathrm{Hom}_A(X, E^3) = 0$ for $\mathrm{gk}(X) \leqslant 2$ (E^3 is pure gk-3), and $\mathrm{Ext}^1_A(X, A) = 0$ for $\mathrm{gk}(X) \leqslant 1$ [ATV2, Sect. 4] we get the result. ∎

Now we look at the exact sequence

$$0 \to E^3/A \to E^2 \to C \to 0, \tag{3.7}$$

where C is defined to the cokernel. In view of (2.18), this sequence can be identified with the sequence

$$0 \to Q/A \to Q/A_{(g)} \oplus Q/\Lambda \to C \to 0. \tag{3.8}$$

We have, therefore, $C \otimes_A \Lambda = 0$, and $C \otimes_A A_{(g)} = 0$. Thus $\mathrm{gk}(C) \leqslant 1$ (Lemma 3.4(ii)).

Lemma 3.9. *E^2 is pure of* gk-*dimension* 2.

Proof. We use the exact sequence (3.7). As $\mathrm{gk}(E^3/A) = 2$, and $\mathrm{gk}(C) \leqslant 1$, we have $\mathrm{gk}(E^2) = 2$. But since E^3/A is pure gk-2 (Lemma 3.6), we see in view of Lemma 3.3 that E^2 is pure gk-2. ∎

Lemma 3.10. *C is pure of* gk-dimension 1.

Proof. We already know that $\mathrm{gk}(C) \leqslant 1$. As $C \neq 0$, it suffices to show that C has no gk-0 submodules or, equivalently, $\mathrm{Hom}_A(k, C) = 0$. We get an exact sequence of cohomology from (3.7),

$$\mathrm{Hom}_A(k, E^2) \to \mathrm{Hom}_A(k, C) \to \mathrm{Ext}^1_A(k, E^3/A),$$

where both the left and right terms are zero: The term on the left is zero because E^2 is pure gk-2. The term on the right is zero because the cohomology sequence of (3.5) gives $\mathrm{Ext}^1_A(k, E^3) = \mathrm{Ext}^2_A(k, A)$, and $\mathrm{Ext}^2_A(k, A) = 0$ be the Gorenstein condition on A [ATV1, 2.11]. ∎

Finally we look at the exact sequence

$$0 \to C \to E^1 \to E^0 \to 0. \tag{3.11}$$

LEMMA 3.12. *E^0 is* gk-0, *and hence pure* gk-0.

Proof. Here we use the fact that E^0 is injective. It is enough to show that E^0 does not contain a finite critical module of gk-dimension greater than 0, for then E^0 would be a direct sum of injective hulls of gk-0 finite critical modules, and therefore would have gk-dimension 0 (1.16)(i). So let X be a finite critical submodule of E^0. Since C is essential in E^1, we have an essential extension $0 \to C \to Y \to X \to 0$, so $\mathrm{Ext}^1_A(X, C) \neq 0$. Considering the cohomology of the sequence (3.7), we see that $\mathrm{Ext}^2_A(X, E^3/A) \simeq \mathrm{Ext}^1_A(X, C)$. The cohomology of the sequence (3.5) gives $\mathrm{Ext}^3_A(X, A) \simeq \mathrm{Ext}^2_A(X, E^3/A)$. Thus we have $\mathrm{Ext}^3_A(X, A) \neq 0$ which means that X has a non-zero finite k-dimensional submodule [ATV2, 2.46]. This, combined with the hypothesis that X is critical, means that X must in fact be a shift of k, so it has gk-dimension 0. ∎

LEMMA 3.13. *E^1 is pure of* gk-*dimension* 1.

Proof. Since C is gk-1 and E^0 is gk-0, the sequence (3.11) tells us that E^1 is gk-1. But since C is pure, the result follows from yet one more application of Lemma 3.3. ∎

We have thus finished the proof of Theorem 3.2.

THEOREM 3.14. *A three-dimensional Sklyanin algebra* (1.13) *is a pure algebra.*

This is a consequence of the purity of the resolution (Theorem 3,2) and the Cohen–Macaulay property of A [ATV2, Le], as the following general remark shows.

(3.15) Let R be an algebra as in (1.1), with $\mathrm{gk}(R) = d > 0$. Assume that R satifies the following conditions:

(1) R is *Cohen–Macaulay* [Le]. This means that for all finite R-modules M we have $\mathrm{gk}(M) + j(M) = d$ where $j(M)$, called the grade number of M, is defined as $j(M) = \min\{i \mid \mathrm{Ext}^i_R(M, R) \neq 0\}$.

(2) The injective dimension of R equals d, and R has a *pure resolution*, i.e., minimal injective resolution of R (in $\mathrm{Gr}-R$ and $R-\mathrm{Gr}$) is of the form

$$0 \to R \to E^d \to E^{d-1} \to \cdots \to E^0 \to 0, \tag{3.16}$$

where E^ν is a pure gk-ν R-module.

Then R is a pure algebra.

To prove the above fact, it suffices to show that every uniform injective R-module does appear, perhaps up to a shift, as a direct factor in one of the terms of the resolution (3.16). Let $E = E_R(M)$ be a uniform injective R-module, where M is a gk-i finite critical R-module ($0 \leqslant i \leqslant d$). It is enough to check that $\mathrm{Hom}_R(M, E^i) \neq 0$, for then there will be a non-zero map from M to some shift of E^i which will necessarily be injective (1.3)(vi), thus implying that $E \subset E^i$, hence E pure of gk-dimension i. Now to see $\mathrm{Hom}_R(M, E^i) \neq 0$, note that $M^\vee = \mathrm{Ext}_R^{d-i}(M, R) = \mathrm{Ext}_R^{j(M)}(M, R) \neq 0$. Now if we compute $M^\vee$ by the injective resolution (3.16) of R we first see that since E^ν are pure gk-ν, we have $\mathrm{Hom}_R(M, E^\nu) = 0$ for $\nu > i$. Thus $\mathrm{Ext}_R^{d-i}(M, R) \hookrightarrow \mathrm{Hom}_R(M, E^i)$, so $\mathrm{Hom}_R(M, E^i) \neq 0$.

4. RESIDUE COMPLEX AND MULTIPLICITIES OF POINT MODULES

We are now ready to describe all the terms in the residue complex for a three-dimensional Sklyanin algebra and, in particular, to compute the multiplicites of the point modules in the complex. We use the notations as in Section 3.

THEOREM 4.1. *Let A be a three-dimensional Sklyanin algebra* (1.13) *and let*

$$0 \to A \to E^3 \to E^2 \to E^1 \to E^0 \to 0 \tag{4.2}$$

be a minimal injective resolution of A in $\mathrm{Gr}-A$ *(resp.* $A-\mathrm{Gr}$*). Then as objects of* $\mathrm{Gr}-A$ *(resp.* $A-\mathrm{Gr}$*):*

(i) $E^3 \simeq Q$,

(ii) $E^2 \simeq Q/A_{(g)} \oplus Q/A$,

(iii) $E^1 \simeq \bigoplus_p E(N_p)$ *where N_p is a right (resp. left) point module, $E(N_p)$ its injective hull, and the direct sum is over all the points of the associated elliptic curve,*

(iv) $E^0 \simeq A'(3)$ *where* $A' = \mathrm{Hom}_k(A, k)$ *is the Matlis-dual module of A.*

Thus the residue comples for a three-dimensional Sklyanin algebra is of the form

$$0 \to A \to Q \to \frac{Q}{A_{(g)}} \oplus \frac{Q}{A} \to \bigoplus_p E(N_p) \to A'(3) \to 0. \tag{4.3}$$

Remark. We see from above that the terms E^0, E^2, E^3 in fact have (A, A)-bimodule structure. Using the idea of localization at orbits of points (introduced in [Aj2]), Amnon Yekutieli has recently shown [Ye3] that one can also realize the E^1 term as a bimodule, thus showing the existence of a bimodule residue complex.

We already know from Proposition 2.18 that (4.1)(i) and (4.1)(ii) are true. Now we prove (4.1)(iv) first, and then (4.1)(iii).

Proof of (4.1)(iv). Up to a shift, the only gk-0 critical A-module is k and its injective hull is A'. As E^0 is pure gk-0 injective (Theorem 3.2), it must be of the form $\bigoplus_i A'(v_i)$ for some integers v_i. We know that $\mathrm{Ext}^3_A(k, A) = k(3)$ [ATV1, 2.11, 2.15]. We now compute $\mathrm{Ext}^3_A(k, A)$ using the *injective* resolution (4.2) of A. As E^j are pure gk-j dimensional (Theorem 3.2), we have $\mathrm{Hom}_A(k, E^j) = 0$ for $j \neq 0$. So the only non-vanishing term in the complex obtained by applying $\mathrm{Hom}_A(k, \ )$ to (4.2) is $\mathrm{Hom}_A(k, E^0) = \mathrm{Hom}_A(k, \bigoplus_i A'(v_i)) = \bigoplus_i k(v_i)$. Thus $\mathrm{Ext}^3_A(k, A) = \bigoplus_i k(v_i)$. Comparing with the known answer we see that v_i equals 3 for only one i and 0 otherwise. This completes the proof. ∎

Proof of (4.1)(iii). First we claim that it is sufficient to show that the k-dimension of $\mathrm{Hom}_A(N_p, E^1)_0$ is 1 for an arbitrary point module N_p. Indeed, E^1 being pure of gk-dimension 1 (Theorem 3.2), the direct factors in the decomposition of E^1 as a direct sum of uniform injective A-modules will all be of class $\mathscr{E}^1$. As the uniform injective A-modules of class $\mathscr{E}^1$ are exactly the injective hulls of the normalized point modules (1.16)(ii), we can write $E^1 = \bigoplus_p E_p^{\mu_p}$ where $E_p^{\mu_p}$ denotes a direct sum of copies of the injective hull $E_p = E_A(N_p)$. Now $\mathrm{Hom}_A(N_p, E_A(N_q))_0 = 0$ for $q \neq p$ (Lemma 4.5). Thus $\mathrm{Hom}_A(N_p, E^1)_0 \simeq \mathrm{Hom}_A(N_p, E_p^{\mu_p})_0$. So given that the k-dimension of $\mathrm{Hom}_A(N_p, E^1)_0$ is 1, μ_p must reduce to 1, thus proving the claim.

Now we show that $\mathrm{Hom}_A(N_p, E^1)_0$ does indeed have k-dimension 1. Note that $\mathrm{Hom}_A(N_p, A) = 0$ [ATV2, 4.3(ii)], and E^j being pure gk-j we have $\mathrm{Hom}_A(N_p, E^j) = 0$ for $j = 2, 3$. Further, using (4.1)(iv),

$$\mathrm{Hom}_A(N_p, E^0) = \mathrm{Hom}_A(N_p, A'(3)) = N'_p(3),$$

where N'_p is the Matlis-dual $\mathrm{Hom}_k(N_p, k)$ of N_p (and *not* the dual point module $N_p^\vee$ of [ATV2]). Thus the complex obtained by applying $\mathrm{Hom}_A(N_p, \ )$ to the injective resolution (4.2) will be of the form

$$0 \to 0 \to 0 \to 0 \to \mathrm{Hom}_A(N_p, E^1) \to N'_p(3) \to 0.$$

Knowing already that $\mathrm{Ext}^2_A(N_p, A) = N_p^\vee$ (the dual point module) and $\mathrm{Ext}^i_A(N_p, A) = 0$ for $i \neq 2$, we get an exact sequence

$$0 \to N_p^\vee \to \mathrm{Hom}_A(N_p, E^1) \to N'_p(3) \to 0. \tag{4.4}$$

Now we consider the Hilbert functions. It is known that $N_p^\vee$ is a shift by 2 of a normalized point module [Aj1, 2.8]. So the Hilbert function $h_{N_p^\vee}(n)$ is 1 for $n \geqslant -2$, and 0 for $n \leqslant -3$. On the other hand, the Hilbert function $h_{N'_p(3)}(n)$ is 1 for $n \leqslant -3$ and 0 for $n \geqslant -2$. Thus $\dim_k(\mathrm{Hom}_A(N_p, E^1))_n = 1$ for all n, and in particular for $n = 0$. This completes the proof. ∎

LEMMA 4.5. *Let N_p, N_q be point modules. Then* $\mathrm{Hom}_A(N_p, E_A(N_q))_0 = 0$ *for $p \neq q$.*

Proof. If there is a non-zero A-homomorphism of degree 0 from N_p to $E_A(N_q)$, then as N_p is gk-1 critical and $E_A(N_q)$ is gk-1 pure (Theorem 3.14), this map must be injective (1.3)(vi). Then the intersection of N_p and N_q inside $E_A(N_q)$ will be non-zero, thus N_p and N_q will be gk-0 equivalent, which is impossible unless $p = q$ (1.16)(ii). ∎

5. COMPARISON WITH THE CLASSICAL CASE

In this section we look at the residue complex of the classical (commutative) polynomial ring in three variables in a way that would allow us to compare with the quantum case. As before, A will denote the Sklyanin case (1.13) of a quantum polynomial ring in three variables and Q_A, Λ_A have the same meaning as in previous sections.

Throughout this section R denotes the commutative polynomial ring $k[x_0, x_1, x_2]$ in three variables (each of degree 1), and Q_R the $\mathbf{Z}$-graded field of fractions of R. A well-known description of finite critical R-modules, which is equivalent to a description of uniform injective R-modules (Proposition 1.11), is as follows. Up to gk-2 equivalence, there is only one gk-3 finite critical R-module which we can take to be R (Corollary 1.12). Up to gk-1 equivalence, the gk-2 finite critical R-modules are exactly $R/\phi R$ where ϕ runs through the set of irreducible homogeneous elements of R of positive degree, modulo a scalar multiple. Up to gk-0 equivalence, the gk-1 finite critical R-modules are exactly the "point modules" $N_p = R/I_p$ where p runs through the set of all closed points of the projective plane $\mathbf{P}_k^2$ and I_p is the ideal of $\{p\}$. The only gk-0 finite critical R-modules are the shifts of the simple module k. A residue complex or minimal injective resolution of R is of the form

$$0 \to R \to Q_R \to \bigoplus_{\phi} Q_R/R_{(\phi)} \to \bigoplus_{p \in \mathbf{P}_k^2} E_R(N_p) \to R'(3) \to 0, \tag{5.1}$$

where the dual module $R' = \mathrm{Hom}_k(R, k)$ is an injective hull of k; $E_R(N_p)$ denotes an injective hull of N_p; $R_{(\phi)}$ denotes the $\mathbf{Z}$-graded localization of R with respect to the prime ideal (ϕ), and $Q_R/R_{(\phi)}$ is an injective hull of $R/\phi R$ (2.12).

Now fix one of the irreducible elements ϕ in (5.1), call it g_R. Define $\Lambda_R = R[g_R^{-1}]$. Note that Λ_R has global dimension 2 and not 1. The key reason why the quantum counterpart $\Lambda_A = A[g^{-1}]$ had global dimension 1 (Proposition 2.18) was that all A-modules of gk-dimension at most one were killed by Λ_A. In the commutative cases, however, there are gk-1 modules which are not killed by Λ_R (for example, point modules N_p for points p not on the curve defined by g_R). As Λ_R has global dimension 2, Q_R/Λ_R is not an injective Λ_R-module, and therefore not an injective R-module (2.6). In fact, if we tensor (5.1) with Λ_R we get an injective resolution for Λ_R as

$$0 \to \Lambda_R \to Q_R \to \bigoplus_{\phi \neq g_R} Q_R/R_{(\phi)} \to \bigoplus_{p \notin g_R} E_R(N_p) \to 0, \tag{5.2}$$

where "$p \notin g_R$" means points p not on the curve defined by g_R.

On the other hand, in the quantum case, we had (2.20)

$$0 \to \Lambda_A \to Q_A \to Q_A/\Lambda_A \to 0. \tag{5.3}$$

Thus we see that there are two *quantum anomalies*: one is that in the quantum case there are no points outside (g), so the analogue of the last term of (5.2) is zero: $\bigoplus_{p \notin (g)} E_A(N_p) = 0$. The other is that the global dimension of the ring Λ (which describes the open complement of the curve defined by g) drops from 2 in the classical case to 1 in the quantum case. These two anomalies are not only interdependent ((1.15), Proposition 2.18) but, in a way, they also cancel each other and so we get a resolution (4.2) of the quantum polynomial ring A with the same multiplicities of the point modules as in the classical case.

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